ELSEVIER

Topology and its Applications 106 (2000) 35–47

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

LS category: product formulas ☆

Paul-Eugène Parent [1]

UCL – Institut de Mathématiques, 2, Chemin du Cyclotron, 1348 Louvain-la-Neuve, Belgium

Received 23 April 1998; received in revised form 14 December 1998

Abstract

Given two continuous maps $f : X \to Y$ and $g : W \to Z$ between simply connected CW-complexes of finite type, we show that $Mcat_{\Bbbk}(f \times g) = Mcat_{\Bbbk} f + Mcat_{\Bbbk} g$ over any field $\Bbbk$. Moreover, we establish a characterization $Mcat_{\Bbbk} f = e_{C^*(Y;\Bbbk)} C_*(X;\Bbbk)$, where e is a kind of Toomer invariant introduced by Félix et al. (1998). Finally, we apply the characterization to show that $Mcat_{\mathbb{Q}} f = Mcat_o f$.

Keywords: LS category; Semifree modules; Semifree extensions

AMS classification: 55P60; 55P62

1. Introduction

Let $f : X \to Y$ be a continuous map between topological spaces. In [1] the authors define the *Lusternik–Schnirelmann* category *cat* f of the map f to be the smallest integer k such that X can be covered by $k+1$ open sets $\{U_i\}$ such that $f|_{U_i}$ is homotopically trivial for all i. In particular they recover the usual *cat* X of some space X as *cat* id_X.

Recall the Ganea construction [7]. Starting with a pathwise connected space X, one considers the loop space fibration

$$\Omega X \overset{j}{\hookrightarrow} PX \overset{\alpha}{\to} X,$$

where $PX = \{(\gamma, r) \mid r \in \mathbb{R}_{\geqslant 0}$ and $\gamma : [0, r] \to X, \gamma(0) = *\}$ and $\alpha(\gamma, r) = \gamma(r)$. Replacing the map j by a cofibration, one obtains an induced map $q_1 : E_1 \to X$ where E_1 is known as the homotopy cofibre of the map j. Then one replaces the map q_1 by a fibration $\alpha_1 : G_1X \to X$. The space $G_1X \simeq E_1$ is known as the first Ganea space. Considering the

☆ This research is supported by a NSERC Scholarship.

[1] Email: parent@agel.ucl.ac.be.

0166-8641/00/$ – see front matter
PII: S0166-8641(99)00072-3

inclusion $j_1 : F_1 \to G_1X$ of the homotopy fibre of the map q_1, one can then iterate this process and obtain the following diagram

$$\begin{tikzcd}
\Omega X \arrow[d, "j"] & F_1 \arrow[d, "j_1"] & & F_k \arrow[d, "j_k"] & \\
PX \arrow[r] \arrow[d, "\alpha"] & G_1X \arrow[r] \arrow[dl, "\alpha_1"] & \cdots \arrow[r] & G_kX \arrow[r] \arrow[dlll, "\alpha_k"] & G_{k+1}X \\
X \arrow[urrr, dotted, bend right, "s"'] & & & &
\end{tikzcd}$$

When X is a normal space, conventional general topology techniques show that *cat* $X \leqslant k$ if and only if the fibration α_k has an homotopy section s as in the above diagram. Similarly, when X and Y are normal spaces, one can show that *cat* $f \leqslant k$ if and only if f factors up to homotopy through the kth Ganea space G_kY of Y.

Considering simply connected CW-complexes of finite type and following the ideas of [4] for spaces, Félix defines in [3] a rational version of *cat* f, i.e., he defines $cat_o\, f$ as *cat* $f_{\mathbb{Q}}$ where $f_{\mathbb{Q}}$ is the induced map between the rational spaces $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$. Moreover, by applying Sullivan's functor A_{pl}, he shows that $cat_o\, f \leqslant k$ if and only if the relative Sullivan model $\bar{f} : \Lambda V \to \Lambda(V \oplus W)$ of f factors through the model $i : \Lambda V \to \Lambda(V \oplus W')$ of the projection $\Lambda V \to \Lambda V/\Lambda^{>k}V$, i.e., there is a *commutative differential graded algebra* (CDGA)-map ρ such that the diagram

$$\begin{tikzcd}
\Lambda V \arrow[r, "\bar{f}"] \arrow[d] \arrow[dr, "i"] & \Lambda(V \oplus W) \\
\Lambda V/\Lambda^{>k}V & \Lambda(V \oplus W') \arrow[l, "\simeq"'] \arrow[u, "\rho"']
\end{tikzcd}$$

commutes in the category of CDGAs. As in [9], he goes a step further by defining $Mcat_o\, f \leqslant k$ if in the above diagram ρ is simply a morphism of ΛV-modules. Clearly we have

$$Mcat_o\, f \leqslant cat_o\, f \leqslant cat\, f.$$

In what will follow, $\Bbbk$ is an arbitrary field unless otherwise specified and all topological spaces are simply connected CW-complexes of finite type. Our aim is to extend the definition of $Mcat_o\, f$ to one over any field $\Bbbk$ and to show

Theorem 2. $Mcat_{\Bbbk}(f \times g) = Mcat_{\Bbbk}\, f + Mcat_{\Bbbk}\, g$.

As an application, Theorem 2 together with the work of Hess [10] give us a unified way of obtaining results already established in [5]. Moreover, we get

Corollary 6. $Acat_{\Bbbk}(X \times Y) = Acat_{\Bbbk}\, X + Acat_{\Bbbk}\, Y$, where $Acat_{\Bbbk} X \leqslant cat\, X$ is another approximation to $cat\, X$ introduced in [9]. We finally show that over $\mathbb{Q}$, the noncommutative invariant $Mcat_{\mathbb{Q}}\, f$ corresponds to the commutative one, i.e.,

Theorem 11. *For any continuous map $f : X \to Y$*

$$Mcat_o\, f = Mcat_{\mathbb{Q}}\, f.$$

2. Definitions

Let $\Bbbk$ be an arbitrary field and let (A, d) be a differential graded algebra (DGA) equipped with a decreasing filtration, I, of differential ideals such that $A = I^0 \supset I^1 \supset \cdots \supset I^k \supset \cdots$ and $I^k \cdot I^l \subset I^{k+l}$.

Let $\phi : (M, d) \to (N, d)$ be a morphism of (A, d)-modules. Recall that a semifree extension of ϕ is a commutative diagram

$$\begin{tikzcd} & N \\ M \arrow[ur, "\phi"] \arrow[r, hook] & P \arrow[u, "\simeq"'] \end{tikzcd}$$

with $P = \bigcup_{i \geqslant -1} P(i)$ where $P(-1) \subset P(0) \subset \cdots$, and each one is an (A, d)-submodule, such that $P(-1) = (M, d)$, and each $P(k)/P(k-1)$, $k \geqslant 0$, is A-free on a basis of cycles. A semifree resolution of an (A, d)-module M is a semifree extension of the map $0 \to M$ [6].

Let $P \xrightarrow{\simeq} M$ be a semifree resolution of M.

Definition 1. A model for ϕ is a semifree extension $\bar{\phi} : P \to Q$ of the composition $P \xrightarrow{\simeq} M \xrightarrow{\phi} N$.

$$\begin{tikzcd} & & N \\ & M \arrow[ur, "\phi"] & \\ P \arrow[ur, "\simeq"] \arrow[rr, "\bar{\phi}"] & & Q \arrow[uu, "\simeq"'] \end{tikzcd}$$

Definition 2. We define an integer $Mcat_A\, \phi$ to be less than or equal to k if there exists a model $\bar{\phi}$ of ϕ which factorizes up to homotopy through a semifree extension of the projection $P \to P/I^{k+1}P$, i.e., in the following diagram

$$\begin{tikzcd} P \arrow[r, "\bar{\phi}"] \arrow[d] & Q \\ P/I^{k+1}P & P' \arrow[l, "\simeq"'] \arrow[u] \end{tikzcd} \tag{1}$$

the top triangle commutes up to homotopy and the bottom one exactly.

Remarks.

- The notion of homotopy is understood to be the one in the differential graded module category, i.e., $f \sim g$ if and only if there exists a map h of degree $-|d|$ such that $f - g = dh + hd$.
- General lifting properties of semifree extensions make the definition of $Mcat_A\,\phi$ independent of the choices of model for ϕ and of the semifree extension of the projection [6].
- If $\phi = id_M$, then $Mcat_A\,\phi = Mcat\,M$ (see [5]).

Let $f : X \to Y$ be a continuous map. Let us consider a free minimal model $(TV, d) \xrightarrow{\simeq} C^*(Y; \mathbb{k})$ of $C^*(Y; \mathbb{k})$ [9], and the filtration on TV given by $I^{k+1} = T^{>k}V$. Then the induced algebra map

$$C^*(Y; \mathbb{k}) \xrightarrow{f^*} C^*(X; \mathbb{k})$$

is in particular a (TV, d)-module morphism and we make the following

Definition 3. $Mcat_{\mathbb{k}}\, f := Mcat_{TV}\, f^*$.

Remarks.

- $Mcat_{\mathbb{k}}\, f$ is independent of the chosen free minimal model of $C^*(Y; \mathbb{k})$ since they are unique up to isomorphism.
- Let $\phi : TV \to T(V \oplus W)$ be a relative extension (in the DGA category, see [6]) of the composition $TV \xrightarrow{\simeq} C^*(Y) \xrightarrow{f^*} C^*(X)$. Then ϕ is a model of f^* in the sense of Definition 1.
- Clearly, we could have considered right TV-modules instead of left TV-modules. But, since $C^*(\cdot; \mathbb{k}) \cong (C^*(\cdot; \mathbb{k}))^{op}$ (see [9]), the two possible values of $Mcat_{\mathbb{k}}\, f$ coincide.

In the particular case of $\mathbb{k} = \mathbb{Q}$ we have another option. We can consider the A_{pl} functor of Sullivan [12] and take the minimal model $(\Lambda V, d)$ of $A_{pl}(Y)$ as our DGA with filtration $I^{k+1} = \Lambda^{>k}V$. Thus we make the following

Definition 4. $Mcat_o\, f := Mcat_{\Lambda V}\, A_{pl}(f)$.

Remarks.

- The integer $Mcat_o\, f$ is well defined since minimal models are unique up to isomorphism.
- The relative Sullivan model $\Lambda V \to \Lambda(V \oplus W)$ of the composition

  $$\Lambda V \xrightarrow{\simeq} A_{pl}(Y) \xrightarrow{A_{pl}(f)} A_{pl}(X)$$

  is a model of $A_{pl}(f)$ in the sense of Definition 1.
- The problem of left ΛV-modules or right ΛV-modules does not pose itself since ΛV is commutative.

3. Product formulas

Recall from [5] the definition of the integer $e_A M$ associated to an (A,d)-module M. One starts by taking a free model (TV,d) of the DGA (A,d) (if it is not already equipped with a desirable decreasing filtration) and considers a (TV,d)-semifree resolution P of M. Then $e_A M$ is the least integer k such that the projection $P \to P/T^{>k}V \cdot P$ is injective in homology.

In the same way, when $\Bbbk = \mathbb{Q}$ we can define $e_{oA}M$ for M a module over some commutative DGA (A,d). But this time we consider the minimal model $(\Lambda V,d)$ of (A,d).

The next result is implicitly given in [5]. For completeness we include its proof. Suppose that $\{(A,d),I\}$ and $\{(B,d),J\}$ are two filtered DGAs and that M and N are, respectively (A,d) and (B,d)-modules. Then

Lemma 1 [5]. $e_{A\otimes B}(M\otimes N) = e_A M + e_B N$.

Proof. Without loss of generality we can consider M and N to be semifree. Clearly $M\otimes N$ is $A\otimes B$-semifree. Let $e_A M = k$ and $e_B N = l$. Then the map

$$M\otimes N \to M/I^{k+1}M \otimes N/J^{l+1}N$$

is injective in homology and factors through the projection

$$M\otimes N \to M\otimes N/(I\otimes J)^{k+l+1}M\otimes N.$$

Thus $e_{A\otimes B}(M\otimes N) \leqslant k+l$. On the other hand, let $\alpha \in M$ and $\beta \in N$ be two non-trivial cocycles killed respectively in $M/I^k M$ and $N/J^l N$. They can be written as $\alpha = d\alpha'' + \alpha'$ and $\beta = d\beta'' + \beta'$ with $d\alpha' = d\beta' = 0$, $\alpha' \in I^k M$ and $\beta' \in J^l N$. Now $\alpha\otimes\beta$ is a non-trivial cocycle in $M\otimes N$ and can be rewritten as

$$\alpha\otimes\beta = d(\alpha''\otimes\beta \pm \alpha'\otimes\beta'') + \alpha'\otimes\beta',$$

where $\alpha'\otimes\beta' \in (I\otimes J)^{k+l}M\otimes N$. It is thus killed in $M\otimes N/(I\otimes J)^{k+l}M\otimes N$, i.e.,

$$e_{A\otimes B}(M\otimes N) \geqslant k+l. \qquad \square$$

Let $f: X\to Y$ and $g: W\to Z$ be two continuous maps. Then

Theorem 2. $\mathit{Mcat}_\Bbbk(f\times g) = \mathit{Mcat}_\Bbbk f + \mathit{Mcat}_\Bbbk g$.

Recall that $C^*(Y\times Z;\Bbbk)$ and $C^*(Y;\Bbbk)\otimes C^*(Z;\Bbbk)$ are quasi-isomorphic as DGA's. The theorem will thus follow from Lemma 1 and the following proposition.

Proposition 3. $\mathit{Mcat}_\Bbbk f = e_{C^*(Y;\Bbbk)} C_*(X;\Bbbk)$.

To prove this proposition, we establish the two inequalities independently.

Lemma 4. *Let M be a $C^*(X;\Bbbk)$-module. Then*

$$e_{C^*(Y;\Bbbk)} M \leqslant \mathit{Mcat}_\Bbbk f.$$

Proof. Let $\phi : TV \to T(V \oplus W)$ be a model of f. This makes M into a $T(V \oplus W)$-module and in particular into a TV-module. Let $P \xrightarrow{\simeq} M$ be a left $T(V \oplus W)$-semifree resolution of M. Note that it is also TV-semifree. Suppose $Mcat_{\Bbbk} f = k$, i.e., there exists a morphism of right TV-modules ψ together with a diagram

$$\begin{tikzcd} TV \arrow[r, "\phi"] \arrow[d] \arrow[dr] & T(V \oplus W) \\ TV/T^{>k}V & T(V \oplus W') \arrow[l, "\simeq"'] \arrow[u, "\psi"'] \end{tikzcd}$$

of the same type as diagram (1). Note that $T(V \oplus W')$ is a free TV-module and can be written as $(\Bbbk \oplus S) \otimes TV$ for some graded vector space S. Now, on one hand, ψ can be extended to a morphism $\bar{\psi} : (\Bbbk \oplus S) \otimes T(V \oplus W) \to T(V \oplus W)$ of $T(V \oplus W)$-modules. On the other hand, if we denote by i the inclusion $T(V \oplus W) \to (\Bbbk \oplus S) \otimes T(V \oplus W)$, we have

$$\begin{tikzcd} P = T(V \oplus W) \otimes_{T(V\oplus W)} P \arrow[r, "i\otimes 1"'] & (\Bbbk \oplus S) \otimes T(V \oplus W) \otimes_{T(V\oplus W)} P \arrow[l, bend right, "\bar{\psi}\otimes 1"'] \end{tikzcd}$$

such that $\bar{\psi} \otimes 1 \circ i \otimes 1 \sim id$. Thus $H^*(i \otimes 1)$ is injective. But the right-hand side of the above diagram can be rewritten as

$$\begin{aligned} (\Bbbk \oplus S) \otimes T(V \oplus W) \otimes_{T(V\oplus W)} P &= (\Bbbk \oplus S) \otimes P \\ &= (\Bbbk \oplus S) \otimes TV \otimes_{TV} P \\ &\simeq TV/T^{>k}V \otimes_{TV} P \\ &= P/T^{>k}V \cdot P, \end{aligned}$$

where the quasi-isomorphism comes from the fact that P is TV-semifree. Thus $H^*(P) \to H^*(P/T^{>k}V \cdot P)$ is injective. $\square$

Lemma 5. *$Mcat_{\Bbbk} f \leqslant e_{C^*(Y;\Bbbk)} C_*(X;\Bbbk)$.*

Proof. Let $\phi : TV \to T(V \oplus W)$ be a model of f. Let $P \xrightarrow{\simeq} C_*(X;\Bbbk)$ be a $T(V \oplus W)$-semifree resolution of $C_*(X;\Bbbk)$. Again note that it is also TV-semifree. Suppose that $e_{C^*(Y;\Bbbk)} C_*(X;\Bbbk) = k$, i.e., $P \xrightarrow{\rho} P/T^{>k}V \cdot P$ is injective in homology. Taking the dual we have

$$T(V \oplus W) \xrightarrow{\simeq} C^*(X;\Bbbk) \xrightarrow{\simeq} P^{\vee}. \tag{2}$$

Let $z = \rho^{\vee}\alpha$ be a cocycle representing $\bar{1} \in H^0(P^{\vee}) \cong H^0(X)$ for some $\alpha \in (P/T^{>k}V \cdot P)^{\vee}$. This is possible since $H^*(\rho^{\vee})$ is onto. Thus the map

$$\begin{aligned} T(V \oplus W) &\to P^{\vee} \\ 1 &\mapsto z \end{aligned}$$

of $T(V \oplus W)$-modules is a quasi-isomorphism. Note that since $T^{>k}V \cdot \alpha \equiv 0$, this map factors through $T(V \oplus W)/T^{>k}V \cdot T(V \oplus W)$. We thus have to show the existence of the TV-module morphism ψ in the following commutative diagram.

$$\begin{tikzcd}
TV \arrow[rr, "\phi"] \arrow[dd] \arrow[dr] & & T(V \oplus W) \arrow[rr, "\simeq"] \arrow[ddr] & & P^\vee \\
& T(V \oplus W') \arrow[ur, dotted, "\psi"'] \arrow[dl, "\simeq"'] & & & \\
TV/T^{>k}V \arrow[rrr] & & & T(V \oplus W)/T^{>k}V \cdot T(V \oplus W) \arrow[uur] &
\end{tikzcd}$$

Note that this diagram can be written in a more familiar way, as shown below.

$$\begin{tikzcd}
T(V \oplus W) \arrow[r, "\simeq"] & P^\vee \\
TV \arrow[u, "\phi"] \arrow[r] & T(V \oplus W') \arrow[u] \arrow[ul, dotted, "\psi"']
\end{tikzcd}$$

Then the existence of ψ is given by the lifting property of semifree extensions. $\square$

Ndombol showed recently in [11] that $Acat_{\Bbbk} X = mcat_{\Bbbk} X$ over any field $\Bbbk$ for any simply connected CW-complex X of finite type. Since $Mcat_{\Bbbk}\, id_X = mcat_{\Bbbk} X$ we have

Corollary 6. $Acat_{\Bbbk}(X \times Y) = Acat_{\Bbbk} X + Acat_{\Bbbk} Y$.

Moreover, if $H^*(X;\Bbbk)$ satisfies Poincaré duality, then $C^*(X;\Bbbk)$ and $C_*(X;\Bbbk)$ are quasi-isomorphic as $C^*(X;\Bbbk)$-modules. Since the Toomer invariant [13] $e_{\Bbbk}(X)$ of a space X is simply $e_{C^*(X;\Bbbk)} C^*(X;\Bbbk)$, we have

Corollary 7. *If X is simply connected and $H^*(X;\Bbbk)$ is a Poincaré duality algebra, then* $Acat_{\Bbbk} X = e_{\Bbbk}(X)$.

Remark. In the last two lemmas, if we consider $A_{pl}(\cdot)$ instead of $C^*(\cdot;\mathbb{Q})$, $(A_{pl}(X))^\vee$ instead of $C_*(X;\mathbb{Q})$, and free commutative algebras, $\Lambda(\cdot)$, instead of free algebras, $T(\cdot)$, then we get a commutative version of Proposition 3, i.e.,

Proposition 8. $Mcat_o\, f = e_{o A_{pl}(Y)}(A_{pl}(X))^\vee$.

Note that since we are working over finite type CW-complexes, the double dual map is a quasi-isomorphism and therefore line (2) in the proof of Lemma 5 is replaced by

$$A_{pl}(X) \xrightarrow{\simeq} (A_{pl}(X))^{\vee\vee} \xrightarrow{\simeq} P^\vee.$$

Now, since $A_{pl}(Y \times Z)$ and $A_{pl}(Y) \otimes A_{pl}(Z)$ are quasi-isomorphic as CDGAs, we have the commutative analogue of Theorem 2, i.e.,

Theorem 9. $Mcat_o(f \times g) = Mcat_o\, f + Mcat_o\, g$.

Hess showed in [10] that $mcat_o\, X = cat_o\, X$ for any simply connected CW-complex X of finite type. Noting that $Mcat_o\, id_X = mcat_o\, X$, we recover the result from [5], i.e.,

Corollary 10 [5]. $cat_o(X \times Y) = cat_o\, X + cat_o\, Y$.

4. Toomer's invariant

For now on we will be working over $\mathbb{Q}$. Sullivan's theory [12] implies the existence, for any simply connected CW-complex X of finite type, of a DGA $Q(X)$ together with a chain

$$A_{pl}(X) \xrightarrow{\simeq} Q(X) \underset{s}{\xleftarrow{\simeq}} C^*(X;\mathbb{Q}) \tag{3}$$

of DGA quasi-isomorphisms natural in X (see [8] for the exact nature of $Q(X)$). Thus any free model $(TZ, d) \xrightarrow{\simeq} C^*(X;\mathbb{Q})$ lifts to a DGA quasi-isomorphism $(TZ, d) \xrightarrow{\simeq} A_{pl}(X)$. Moreover, from any continuous map $f : X \to Y$, one constructs a commutative diagram

$$\begin{tikzcd}
A_{pl}(Y) \arrow[r, "A_{pl}(f)"] & A_{pl}(X) \\
TZ \arrow[u, "\simeq"] \arrow[d, "\simeq"'] \arrow[r, dotted, "\phi"] & T(Z \oplus W) \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \\
C^*(Y;\mathbb{Q}) \arrow[r, "f^*"] & C^*(X;\mathbb{Q})
\end{tikzcd}$$

that commutes up to homotopy, where ϕ is a relative extension [6] obtained by general lifting arguments from f^*. In particular we have

$$\begin{aligned}
e_{C^*(Y)} C_*(X) &= e_{TZ}\big(C^*(X)\big)^\vee \\
&= e_{TZ} T(Z \oplus W)^\vee \\
&= e_{A_{pl}(Y)}\big(A_{pl}(X)\big)^\vee.
\end{aligned}$$

This leads to

Theorem 11. *For any continuous map $f : X \to Y$*

$$Mcat_o\, f = e_{o\,A_{pl}(Y)}\big(A_{pl}(X)\big)^\vee = e_{A_{pl}(Y)}\big(A_{pl}(X)\big)^\vee = e_{C^*(Y)} C_*(X) = Mcat_{\mathbb{Q}}\, f.$$

We show the second equality via the following proposition.

Proposition 12. *Let M be an $A_{pl}(X)$-module for some space X. Then*

$$e_{A_{pl}(X)} M = e_{o\,A_{pl}(X)} M.$$

Proof. Without loss of generality we can assume M to be ΛV-semifree where $(\Lambda V, d)$ is the minimal model of X. From the above remarks, M is thus a TZ-module. Let $\Phi : P \xrightarrow{\simeq} M$ be a TZ-semifree resolution of M. Then the commutative diagram

$$\begin{tikzcd}
P \arrow[rr, "\Phi", "\simeq"'] \arrow[d] & & M \arrow[dl] \arrow[dr] & \\
P/T^{>k}Z \cdot P \arrow[r] & M/T^{>k}Z \cdot M \arrow[rr] & & M/\Lambda^{>k}V \cdot M
\end{tikzcd}$$

shows that $e_{TZ}M \leqslant e_{o\Lambda V}M$. We proceed to show the reverse inequality.

Let G_nX denote the nth Ganea space. Doeraene [2] showed that $TZ/T^{>n}Z$ and $C^*(G_nX)$ were quasi-isomorphic as TZ-modules through a chain of quasi-isomorphisms of the form

$$TZ/T^{>n}Z \xleftarrow{\simeq} (\mathbb{Q} \oplus S) \otimes TZ \xrightarrow{\simeq} C^*(G_nX)$$

for some graded vector space S. We can thus construct a commutative diagram

$$\begin{tikzcd}
P \arrow[rr, "\simeq"] \arrow[d] \arrow[dr, "i"] & & C^*(X) \otimes_{TZ} P \arrow[d, "\alpha_n^* \otimes 1"] \\
P/T^{>n}Z \cdot P & (\mathbb{Q} \oplus S) \otimes P \arrow[l, "\simeq"'] \arrow[r, "\simeq"'] & C^*(G_nX) \otimes_{TZ} P
\end{tikzcd}$$

where $\alpha_n : G_nX \to X$ is the nth Ganea's fibration and i is the unique TZ-module morphism that sends p to $1 \otimes p$. Note that the quasi-isomorphisms are preserved because P is TZ-semifree. Moreover, the projection $P \to P/T^{>n}Z \cdot P$ is injective in homology if and only if $\alpha_n^* \otimes 1$ is. Now, from the naturality of (3), we construct a commutative diagram

$$\begin{tikzcd}
C^*(X) \otimes_{TZ} P \arrow[r, "\alpha_n^* \otimes 1"] \arrow[d, "s_1 \otimes \Phi"'] & C^*(G_nX) \otimes_{TZ} P \arrow[d, "s_2 \otimes \Phi"] \\
Q(X) \otimes_{\Lambda V} M \arrow[r] & Q(G_nX) \otimes_{\Lambda V} M \\
A_{pl}(X) \otimes_{\Lambda V} M \arrow[r, "A_{pl}(\alpha_n) \otimes 1"] \arrow[u, "\simeq"] & A_{pl}(G_nX) \otimes_{\Lambda V} M \arrow[u, "\simeq"']
\end{tikzcd}$$

in which if $s_1 \otimes \Phi$ and $s_2 \otimes \Phi$ were quasi-isomorphisms, then $\alpha_n^* \otimes 1$ would be injective in homology if and only if $A_{pl}(\alpha_n) \otimes 1$ was. Note that in the bottom square the quasi-isomorphisms are preserved because M is ΛV-semifree. Before showing that $s_1 \otimes \Phi$ and $s_2 \otimes \Phi$ are indeed quasi-isomorphisms, we need to recall basic constructions of a semifree resolution of some R-module N.

Proposition 13 [6]. *Every R-module N has a semifree resolution M.*

Proof. Let $M(0) = R \otimes (S(0), 0)$ where $S(0)$ is the free $\Bbbk$-module generated by the elements of the submodule of cycles of N. The linear injection $S(0) \to N$ extends to $M(0)$ as an R-module morphism $q(0)$. Clearly $H_*(q(0))$ is surjective. We now construct inductively a sequence of maps $q(k) : M(k) \to N$. Suppose $q(k-1)$ constructed and let $\{v_{\alpha_i}\}$ be a basis in degree i of $S(k)$ in one-to-one correspondence with the elements $\{w_{\alpha_i}\}$ of *ker* $H_{i-1}(q(k-1))$. Set $dv_{\alpha_i} = w_{\alpha_i}$ and let $M(k) = M(k-1) \oplus (R \otimes S(k))$. Since $H_{i-1}(q(k-1))w_{\alpha_i} = 0$, we have $q(k-1)(w_{\alpha_i}) = dx_{\alpha_i}$. Now extend the linear map $v_{\alpha_i} \mapsto x_{\alpha_i}$ to $R \otimes S(k)$ as an R-module morphism. This morphism together with $q(k-1)$ define $q(k)$. Thus

$$q = \varinjlim q(k) : M = \varinjlim M(k) \xrightarrow{\simeq} N$$

is clearly the required semifree resolution. $\square$

Proposition 14. *Let $\phi : R' \xrightarrow{\simeq} R$ be a DGA quasi-isomorphism. Suppose M is R-semifree of the same format as in Proposition* 13. *Then there is a semifree R'-module $P = \bigcup P(k)$, where $P(0) = R' \otimes S(0)$ and $P(k) = P(k-1) \oplus (R' \otimes S(k))$, together with a quasi-isomorphism $\Phi = \varinjlim \Phi(k) : P \to M$, where $\Phi(0) = \phi \otimes id_{S(0)}$, such that for $v \in S(k)$, we have $\Phi(v) - v \in M(k-1)$.*

Proof. We construct $\Phi(k)$ and $d'|_{S(k)}$ by induction on k. Clearly $\Phi(0)$ is a quasi-isomorphism on $(P(0), d') = R' \otimes (S(0), 0)$. Assume that d' is constructed up to $P(k-1)$ with $\Phi(k-1)$ a quasi-isomorphism. Now, for $v_{\alpha_i} \in S(k)$, dv_{α_i} is a non-trivial cycle in $M(k-1)$ by construction. Thus, since $\Phi(k-1)$ is a quasi-isomorphism, there is a non-trivial cycle $p_{\alpha_i} \in P(k-1)$ such that $\Phi(k-1)(p_{\alpha_i}) = dv_{\alpha_i} + d\gamma_{\alpha_i}$, $\gamma_{\alpha_i} \in M(k-1)$. Let $d'v_{\alpha_i} = p_{\alpha_i}$ and $\Phi(k)(v_{\alpha_i}) = v_{\alpha_i} + \gamma_{\alpha_i}$. Clearly $d \circ \Phi(k) = \Phi(k-1) \circ d'$. Now the following commutative diagram between short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & P(k-1) \arrow[r] \arrow[d, "\Phi(k-1)"', "\simeq"] & P(k-1) \oplus (R' \otimes S(k)) \arrow[r] \arrow[d, "\Phi(k)"'] & R' \otimes (S(k), 0) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & M(k-1) \arrow[r] & M(k-1) \oplus (R \otimes S(k)) \arrow[r] & R \otimes (S(k), 0) \arrow[r] & 0
\end{tikzcd}$$

together with its induced long exact sequence in homology and the "five lemma" show that $\Phi(k)$ is a quasi-isomorphism. $\square$

Proof of Proposition 12 (*Continued*). Using the same notation as in the last two propositions, we can assume that M and P are of the given format. We are now in a position to show that $s_1 \otimes \Phi$ and $s_2 \otimes \Phi$ are quasi-isomorphisms. Note that

$$\begin{tikzcd}
C^*(\cdot) \otimes S(0) = C^*(\cdot) \otimes_{TZ} P(0) \arrow[d, "s_i \otimes \Phi(0)"] \\
Q(\cdot) \otimes S(0) = Q(\cdot) \otimes_{\Lambda V} M(0)
\end{tikzcd}$$

is a quasi-isomorphism for respectively $\cdot = X,\ G_nX$ and $i = 1, 2$. Proceeding by induction, we suppose that $s_i \otimes \Phi(k-1)$ is a quasi-isomorphism. Again the following commutative diagram between short exact sequences

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
C^*(\cdot)\otimes_{TZ} P(k-1) \arrow[r, "s_i\otimes\Phi(k-1)", "\simeq"'] \arrow[d] & Q(\cdot)\otimes_{\Lambda V} M(k-1) \arrow[d] \\
C^*(\cdot)\otimes_{TZ} P(k-1) \oplus (C^*(\cdot)\otimes S(k)) \arrow[r, "s_i\otimes\Phi(k)"] \arrow[d] & Q(\cdot)\otimes_{\Lambda V} M(k-1) \oplus (Q(\cdot)\otimes S(k)) \arrow[d] \\
C^*(\cdot)\otimes(S(k), 0) \arrow[r, "\simeq"'] \arrow[d] & Q(\cdot)\otimes(S(k), 0) \arrow[d] \\
0 & 0
\end{tikzcd}$$

together with its induced long exact sequence in homology and the "five lemma" show that $s_i \otimes \Phi(k)$ is a quasi-isomorphism for respectively $\cdot = X, G_nX$ and $i = 1, 2$. Thus the result.

To conclude, we need the following lemma.

Lemma 15. *The relative Sullivan model $p_n : \Lambda V \to \Lambda(V \oplus E_n)$ of the nth Ganea fibration α_n factors through the relative Sullivan model $q_n : \Lambda V \to \Lambda(V \oplus V_n)$ of the projection $\Lambda V \to \Lambda V/\Lambda^{>n}V$, i.e.,*

$$\begin{tikzcd}
\Lambda V \arrow[rr, "p_n"] \arrow[dr, "q_n"'] & & \Lambda(V\oplus E_n) \\
& \Lambda(V\oplus V_n) \arrow[ur, "\psi"'] &
\end{tikzcd}$$

with $\psi \circ q_n = p_n$ for some DGA map ψ.

Proof. Recall from [3] that $cat_0(G_nX) \leqslant n$. Moreover its model must be a retract of some CDGA A with $A^{n+1} = 0$ [3, p. 44]. We thus have a commutative diagram

$$\begin{tikzcd}
& \Lambda(V\oplus V_n) \arrow[d, "\simeq"] \arrow[ddd, bend left=60, dotted, "\tau"] \\
\Lambda V \arrow[ur, "q_n"] \arrow[r] \arrow[d, "p_n"'] & \Lambda V/\Lambda^{>n}V \arrow[d] \\
\Lambda(V\oplus E_n) \arrow[r] & A \\
& \Lambda T \arrow[u, "\simeq"'] \arrow[ul, "\rho"]
\end{tikzcd}$$

where τ is obtained through standard lifting arguments on relative Sullivan models. The desired map is given by $\psi = \rho \circ \tau$. $\square$

Proof of Proposition 12 (*Continued*). Using the notation of the last lemma, we have a commutative diagram

$$\begin{tikzcd}
A_{pl}(X) \otimes_{\Lambda V} M \arrow[r, "A_{pl}(\alpha_n)\otimes 1"] & A_{pl}(G_n X) \otimes_{\Lambda V} M \\
M = \Lambda V \otimes_{\Lambda V} M \arrow[u, "\simeq"] \arrow[r, "p_n\otimes 1"] \arrow[dr, "q_n\otimes 1"'] \arrow[ddr, bend right] & \Lambda(V \oplus E_n) \otimes_{\Lambda V} M \arrow[u, "\simeq"'] \\
& \Lambda(V \oplus V_n) \otimes_{\Lambda V} M \arrow[u, "\psi\otimes 1"'] \arrow[d, "\simeq"] \\
& M/\Lambda^{>n} V \cdot M
\end{tikzcd}$$

which shows that the projection $M \to M/\Lambda^{>n} V \cdot M$ is injective in homology if $A_{pl}(\alpha_n) \otimes 1$ is, i.e., if the projection $P \to P/T^{>n} Z \cdot P$ is. Again, note that the quasi-isomorphisms are preserved since M is ΛV-semifree. The result follows. $\square$

Note that we also recover a result from [5], i.e.,

Corollary 16 [5].

$$cat_0\, X = e_{C^*(X;\mathbb{Q})} C_*(X;\mathbb{Q}).$$

References

[1] I. Berstein, P. Ganea, The category of a map and of a cohomology class, Fund. Math. 50 (1961) 261–279.

[2] J.-P. Doeraene, LS-category in a model category, J. Pure Appl. Algebra 84 (1993) 215–261.

[3] Y. Félix, La dichotomie elliptique-hyperbolique en homotopie rationnelle, Soc. Math. de France, Astérisque 176 (1989).

[4] Y. Félix, S. Halperin, Rational L.S. category and its applications, Trans. AMS 273 (1982) 1–37.

[5] Y. Félix, S. Halperin, J.-M. Lemaire, The rational LS category of products and of Poincaré duality complexes, Topology 37 (4) (1998) 749–756.

[6] Y. Félix, S. Halperin, J.-C. Thomas, Differential graded algebras in topology, in: Handbook of Algebraic Topology, Elsevier, Amsterdam, 1995, pp. 829–865.

[7] T. Ganea, Lusternik–Schnirelmann category and strong category, Illinois J. Math. 11 (1967) 417–427.

[8] S. Halperin, Lectures on minimal models, Mém. Soc. Math. France (Nouvelle série) 9/10 (1983).

[9] S. Halperin, J.-M. Lemaire, Notions of category in differential algebra, Lecture Notes in Math. 1328, Springer, Berlin, 1988, pp. 138–154.

[10] K. Hess, A proof of Ganea's conjecture for rational spaces, Topology 30 (1991) 205–214.

[11] B. Ndombol, L'invariant *Acat* des algèbres quasi-commutatives, Bull. Belg. Math. Soc. 4 (1997) 227–288.
[12] D. Sullivan, Infinitesimal computations in topology, Publ. I.H.E.S. 47 (1977) 269–331.
[13] G.H. Toomer, Lusternik–Schnirelmann category and the Moore spectral sequence, Math. Z. 138 (1974) 123–143.